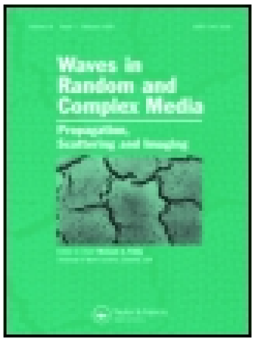

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Reflection of thermoelastic waves from the isothermal boundary of a solid half-space under memory-dependent heat transfer

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ABSTRACT

In this paper, we investigated the reflection of thermoelastic plane waves from the isothermal stress-free boundary of a homogeneous, isotropic and thermally conducting solid half-space in the context of the new linear theory of generalized thermoelasticity under heat transfer with memory-dependent derivative. It has been found that three types of basic waves consisting of two sets of coupled longitudinal waves and one independent vertically shear type wave may travel with distinct phase speeds. The formulae for various reflection coefficients are determined in case of an incident coupled dilatational elastic wave at an isothermal stress-free boundary of the medium. For an appropriate material, the reflection coefficients are computed numerically and presented graphically for various values of the angle of incidence and discussed the effect of various parameters of interest. At the end, the phase speeds and the attenuations coefficients of the coupled longitudinal waves are shown graphically to compare our results with the existing results.

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Memory-dependent derivative; plane waves; dispersion; phase speeds and attenuation; reflection

1. Introduction

Scott Blair's model [1], which is basically a material model, includes a formula for memory phenomena in various disciplines. The model takes the form

$$ {}^{0}\mathcal{D}_t^{\alpha}\epsilon(t) = \kappa\sigma(t), \tag{1} $$

where ${}^{0}\mathcal{D}_t^{\alpha}\epsilon(t)$ denotes the fractional-order derivative which depends on the strain history from 0 to t. For integral value of $\alpha = n$, ${}^{0}\mathcal{D}_t^{\alpha}\epsilon(t) = \mathrm{d}^n\epsilon(t)/\mathrm{d}t^n$, and $\kappa > 0$ is a constant. Equation (1) works not only in modeling viscoelastic materials, but also in modeling biological kinetics with memory.

A fractional-order derivative is a generalization of an integer order derivative and integral. It originated from a letter of L'Hopital to Leibnitz in 1695 regarding the meaning of the half-order derivative. The kernel function of a fractional derivative is termed the memory function, but it does not replicate any physical process. Imprecise physical meaning

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has been a big obstacle that keeps fractional derivatives lagging far behind the integer-order calculus. There are several definitions of a fractional derivative. The Riemann–Liouville derivative is one of the most standard definitions

$$ {}^{0}\mathcal{D}_t^{\alpha}\epsilon(t) = \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_0^t \frac{\epsilon(s)}{(t-s)^{1+\alpha-n}}\,\mathrm{d}s, \quad n-1 \le \alpha < n, $$

where $\Gamma(\cdot)$ is the Euler's gamma function and n is an integer. A memory process generally consists of two stages: the first is short, with permanent retention at the beginning, and it cannot be neglected in general, and the second is governed by the fractional model Equation (1). The critical point between the fresh stage and the working stage is usually not the origin. This is quite different from the traditional fractional models of one stage. The key point is that the order of a fractional derivative is an index of memory. The dimensionless form of the solution of Equation (1) is

$$ E(\eta) = \eta^{\alpha} - (\eta - 1)^{\alpha}, \tag{2} $$

where $\eta = t/t_M$ and $E(\eta) = \epsilon(t)/\epsilon_M$, where ϵ_M is the strain at the end of time of creeping $t = t_M$. Equation (2) reveals that $E(\eta)$ increases with an increase in α. The higher the value of the index α, the slower is the forgetting during the process. In particular, at $\alpha = 0$, $E = 0$, meaning that 'nothing is memorized', and $E = 1$ for $\alpha = 1$ which means that 'nothing is forgotten'. Therefore, the fractional order α is basically termed as the index of the memory effect.

For a standard creep and recovery process, the specimen is usually loaded under a constant stress $\sigma(t) = \sigma_0$ from 0 to t_M, and the load is removed at the instant $t = t_M$, then $\sigma(t) = 0$ for $t \ge t_M$. If $H(t)$ is the Heaviside function, Equation (1) takes the following form:

$$ {}^{0}\mathcal{D}_t^{\alpha}\epsilon(t) = \kappa\sigma_0\left(H(t) - H(t-t_M)\right), $$

where ${}^{0}\mathcal{D}_t^{\alpha}\epsilon(t)$ is the Riemann–Liouville fractional-order derivative with zero initial condition. The superposition method gives the solution of the above equation as follows:

$$ \epsilon(t) = \frac{\kappa\sigma_0}{\Gamma(1+\alpha)}\left[t^{\alpha}H(t) - (t-t_M)^{\alpha}H(t-t_M)\right]. $$

This is in agreement with the early observation of the behavior of some viscoelastic materials.

The non-integral (fractional)-order derivatives and the fractional differential equations have gained considerably more attention in the fields of applied sciences and various engineering disciplines [1]. Gorenflo et al. [2], and Atanackovic et al. [3] provided diverse theoretical advances and recent applications of fractional calculus. One hindrance to the wider use of fractional-order methods by engineers is the absence of a simple geometric picture for the fractional-order integral. There are several definitions of fractional derivatives (e.g. Riemann–Liouville, Caputo, Reisz, and Grunwald–Letnikov [2]), each of which has specific advantages and limitations, particularly when used to define a distribution of fluxes into a control volume or the effects of fading memory on the forces applied in a free body diagram. Diethelm [4] incorporated a kernel function and modified a Caputo-type fractional-order derivative as

$$\mathcal{D}_a^{\alpha} f(t) = \int_a^t k_\alpha(t-\xi) f^{(m)}(\xi)\,\mathrm{d}\xi,$$

where $k_\alpha(t-\xi)$ is the kernel function, and $f^{(m)}$ is the mth order derivative. In applications, $k_\alpha(t-\xi)$ takes some specific form, e.g.

$$k_\alpha(t-\xi) = \frac{(t-\xi)^{m-\alpha-1}}{\Gamma(m-\alpha)}.$$

Wang and Li [5] proposed another form of the fractional derivative with arbitrary kernel $K(t-\xi)$ (can be chosen freely) over a slipping interval $[t-\tau, t]$ as follows:

$$\mathcal{D}_\tau^{(1)} f(t) = \frac{1}{\tau}\int_{t-\tau}^t K(t-\xi) f'(\xi)\,\mathrm{d}\xi, \tag{3}$$

where τ (>0) is called the delay time, which can also be chosen freely. The preceding modifications of fractional-ordered derivatives are termed memory-dependent derivatives. In general, the mth-order memory-dependent derivative of a differentiable function $f(t)$ relative to the time delay, $a > 0$ is defined as

$$\mathcal{D}_a^{(m)} f(t) = \frac{1}{\tau}\int_{t-a}^t K(t,\xi) f^{(m)}(\xi)\,\mathrm{d}\xi,$$

where the time delay a denotes the memory scale, and the kernel function $K(t.\xi)$ must be a differentiable function with respect to its arguments. The kernel function and the memory scales must be chosen in such a way that they are compatible with the physical problem, so this type of derivative provides more possibilities to capture the material response [5]. Generally, the memory effect needs weight $0 \leq K(t-\xi) \leq 1$ for $\xi \in [t-\tau, t]$ so that the magnitude of $D_\tau f(t)$ is usually smaller than that of the common derivative $f'(t)$. Simply the right hand side of (3) is a weighted mean of $f'(t)$. As $\xi \in [t-\tau, t]$, one can easily understand that the function $f(\xi)$ takes value from different points on the time interval $[t-\tau, t]$. Considering our present time as t, we can say $[t-\tau, t)$ is the past time interval. Thus we conclude the main feature of MDD, that is the functional value in real time depends on the past time also. That is why D_τ is called the non-local operator whereas integer order derivative (or integration) is a local operator (i.e. it does not depend on the past time). The kernel function $K(t-\xi)$ can be chosen freely, such as $1, [1-(t-\xi)], [1-(t-\xi)/\tau]^p$ for any positive real number p which may be more practical [5]. They are a monotonic increasing function from 0 to 1 in the interval $[t-\tau, t]$. According to the nature of the problem, one can select a suitable kernel function of his/her choice.

From the Maxwell–Cattaneo theory [6] to Green–Naghdi generalized thermoelasticity models [7], it is well established that the thermal memory has a significant role in the theory of generalized thermoelasticity [8–11]. In the twenty-first century, memory components have been introduced in terms of fractional-order derivatives in numerous forms, see [12–14] for details. In these fractional models of modified heat flux laws, the memory response is described by the fractional index parameter. The memory-dependent derivative (MDD) were first incorporated in the Fourier's law of heat conduction [6], a new hyperbolic-type heat conduction equation, by Wang and Li [5]. This new generalization of hyperbolic-type heat conduction models is accepted as the modified heat conduction law with measuring memory. Following the work of Wang and Li [5], Yu et al. [15] introduced MDD in the heat conduction law as

$$(1+\tau\mathcal{D}_a)\,q_i=-K_T\Theta_{,i}, \tag{4}$$

where $D_a f(t)=D_a^{(1)}f(t)$.

Later, Ezzat et al. [16,17] introduced the first-order MDD into the rate of heat flux in the L-S theory [8] to denote memory-dependence as

$$\left(1+\tau_0\mathcal{D}_{\tau_0}\right)q_i=-K_T\Theta_{,i}, \tag{5}$$

where τ_0 is introduced as the delay time parameter. Equations (4) and (5) provide the following advantages compared with the aforementioned amendments of Fourier's law by using fractional derivatives: (1) the influence of memory dependency claims its superiority in terms of memory scale parameter; (2) in a limiting sense, this simplification develops the Lord-Shulman model of generalized thermoelasticity; and (3) because the kernel function and the memory scale parameters may be chosen subjectively, it is more malleable in many practical applications. Some recent works on generalized thermoelasticity with MDD can be found in the literatures [18–20].

Wave propagation and wave reflection phenomena are applicable in various fields like geophysical exploration, mineral and oil exploration, seismology, etc. The body wave propagation in thermoelastic solids is applicable in various fields of engineering. Several problems on plane harmonic wave propagation in coupled and generalized thermoelasticities have been investigated by many authors during the last five decades. Some of the notable works among them are found in the literatures [21–30]. Recently, Sarkar et al. [31] studied the memory response in plane wave reflection in generalized magneto-thermoelasticity. In the present contribution, we investigate the reflection of thermoelastic plane waves from the isothermal stress-free boundary of a homogeneous, isotropic and thermally conducting solid half-space in the context of the new linear theory of generalized thermoelasticity under heat transfer with memory-dependent derivative [14,16]. It has been found that three types of basic waves consisting of two sets of coupled longitudinal waves and one independent vertically shear wave may travel with distinct phase speeds. The formulae for various reflection coefficients are determined in case of an incident coupled dilatational elastic wave at an isothermal stress-free boundary of the medium considered. For an appropriate material, the reflection coefficients are computed numerically and presented graphically for various values of the angle of incidence and discussed the effect of various parameters of interest. At the end, the phase speeds and the attenuations coefficients of the coupled longitudinal waves are shown graphically to compare our results with the existing results.

2. Governing equations and formulation of the problem

The basic governing equations for a homogeneous, isotropic and thermally conducting elastic solid in the context of the generalized thermoelasticity with memory-dependent derivative heat transfer proposed by Ezzat et al. [14,16], in absence of heat sources and body forces in general Cartesian coordinates system (x, y, z) are:

2.1. Stress–strain–temperature relation

$$\tau_{ij} = 2\mu e_{ij} + (\lambda e - \gamma\Theta)\,\delta_{ij}, \tag{6}$$

where $i, j = x, y, z$, $e_{ij} = (u_{i,j} + u_{j,i})/2$ are the components strain tensor, τ_{ij} and are the components of the stress tensor, $e = u_{i,i}$ is the cubical dilatation, u_i are the displacement components, λ, μ are Lame' constants, $\gamma = (3\lambda + 2\mu)\alpha_T$ is the thermoelastic coupling parameter and α_T is the coefficient of volume expansion.

2.2. Equation of motion

$$\mu u_{i,jj} + (\lambda + \mu)u_{j,ij} - \gamma\Theta_{,i} = \rho\ddot{u}_i, \tag{7}$$

where ρ is the mass density.

2.3. Heat conduction equation with memory-dependent derivative

The classical Fourier's law of heat conduction relates the heat flux vector to the temperature gradient, which can be written in component form as

$$q_i = -K_T\Theta_{,i}, \tag{8}$$

where q_i are the components of the heat flux vector.

The energy equation is read from [32] as

$$\rho C_E\dot{\Theta} + \gamma T_0\dot{e} = -q_{i,i}, \tag{9}$$

where C_E is the specific heat at constant strain.

From a mathematical viewpoint, Ezzat et al. [14,16] and Yu et al. [15] modified the Fourier's law (8) in the theory of generalized heat conduction with memory-dependent derivative having a time-delay parameter τ as

$$(1 + \tau D_\tau)\,q_i = -K_T\Theta_{,i}. \tag{10}$$

Taking the memory-dependent derivative with respect to t of Equation (9), we obtain

$$D_\tau\left(\rho C_E\dot{\Theta} + \gamma T_0\dot{e}\right) = -D_\tau q_{i,i}. \tag{11}$$

Multiplying Equation (11) by the time-delay τ and then adding to Equation (9), we get

$$(1 + \tau D_\tau)\left(\rho C_E\dot{\Theta} + \gamma T_0\dot{e}\right) = -(1 + \tau D_\tau)\,q_{i,i}. \tag{12}$$

Inserting from Equation (10) into the above equation, we have

$$K_T\theta_{,ii} = (1 + \tau D_\tau)\left(\rho C_E\dot{\Theta} + \gamma T_0\dot{e}\right). \tag{13}$$

The above equation can also be written using the definition of memory-dependent derivative as

$$K_T\Theta_{,ii} = \left(\rho C_E\dot{\Theta} + \gamma T_0\dot{e}\right) + \int_{t-\tau}^{t} K(t-\xi)\left(\rho C_E\frac{\partial^2\Theta}{\partial\xi^2} + \gamma T_0\frac{\partial^2 e}{\partial\xi^2}\right)\mathrm{d}\xi. \tag{14}$$

Equation (13) or (14) is the generalized heat transport equation with the memory-dependent derivative having τ as the time-delay. The dynamic coupled theory of heat

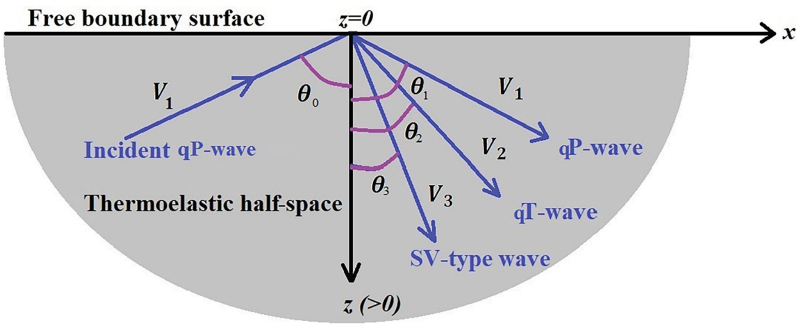

Figure 1. Schematic of the incident and reflected thermoelastic waves at a surface $z = 0$.

conduction law follows as the limit case when $\tau \to 0$. Note that in the above equations, a comma followed by a suffix denotes *spatial derivative* and a *superposed dot* stands for time-differentiation.

We consider a linear homogeneous, isotropic and thermally conducting elastic medium occupying the half-space:

$$\Omega = \{(x, y, z) : -\infty < x, y < \infty, 0 \leq z < \infty\}.$$

Let the origin O of the rectangular Cartesian coordinate system $Oxyz$ be fixed at a point on the boundary $z = 0$ with the z-axis directed normally inside the medium and the $x-$axis is directed along the horizontal direction (see Figure 1). The y-axis is taken in the direction of the line of intersection of the plane wave front with the plane surface. If we restrict our analysis to a plane strain problem parallel to the x–z plane, then the field variables may be taken as functions of x , z and t only. Hence, the displacement components may take the form

$$u_1 = u(x, z, t),\ u_2 = v(x, z, t) = 0,\ u_3 = w(x, z, t).$$

Then, Equations (6), (7) and (13) are simplified to

$$\tau_{xx} = 2\mu \frac{\partial u}{\partial x} + \lambda e - \gamma \Theta, \tag{15}$$

$$\tau_{zz} = 2\mu \frac{\partial w}{\partial z} + \lambda e - \gamma \Theta, \tag{16}$$

$$\tau_{xz} = \mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right), \tag{17}$$

$$\mu \nabla^2 u + (\lambda + \mu) \frac{\partial e}{\partial x} - \gamma \frac{\partial \Theta}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2}, \tag{18}$$

$$\mu \nabla^2 w + (\lambda + \mu) \frac{\partial e}{\partial z} - \gamma \frac{\partial \Theta}{\partial z} = \rho \frac{\partial^2 w}{\partial t^2}, \tag{19}$$

$$K_T \nabla^2 \Theta = \frac{\partial}{\partial t} (1 + \tau D_\tau) (\rho C_E \Theta + \gamma T_0 e) \,. \tag{20}$$

In our present study, we shall deal with the following kernel function

$$K(t-\xi) = A + B(t-\xi) = \begin{cases} \frac{1}{2} & \text{if } A = \frac{1}{2},\ B = 0, \\ \frac{1}{2} - \left(\frac{t-\xi}{\tau}\right) & \text{if } A = \frac{1}{2}, B = -\frac{1}{\tau}, \\ 1 - (t-\xi) & \text{if} A = 1,\ B = -1, \end{cases} \tag{21}$$

where A and B are constants.

To transform the above equations in non-dimensional forms, we define the following non-dimensional variables

$$(x', z') = C_L \eta (x, z), \quad (u', w') = C_L \eta (u, w), \quad t' = C_L^2 \eta t, \quad \Theta' = \frac{\gamma \Theta}{\rho C_L^2}, \quad \sigma'_{ij} = \frac{\sigma_{ij}}{\rho C_L^2},$$

where $C_L^2 = (\lambda + 2\mu)/\rho$ is the speed of classical longitudinal (dilatational) wave and $\eta = \rho C_E / K_T$ is the thermal viscosity. Introducing the above parameters in Equations (15)–(20) and suppressing the primes for convenience, we obtain

$$\tau_{xx} = 2\beta \frac{\partial u}{\partial x} + (1 - 2\beta) e - \Theta, \tag{22}$$

$$\tau_{zz} = 2\beta \frac{\partial w}{\partial z} + (1 - 2\beta) e - \Theta, \tag{23}$$

$$\tau_{xz} = \beta \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right), \tag{24}$$

$$\beta \nabla^2 u + (1 - \beta) \frac{\partial e}{\partial x} - \frac{\partial \Theta}{\partial x} = \frac{\partial^2 u}{\partial t^2}, \tag{25}$$

$$\beta \nabla^2 w + (1 - \beta) \frac{\partial e}{\partial z} - \frac{\partial \Theta}{\partial z} = \frac{\partial^2 w}{\partial t^2}, \tag{26}$$

$$\nabla^2 \Theta = \frac{\partial}{\partial t} (1 + \tau D_\tau) (\Theta + \varepsilon e), \tag{27}$$

where $\nabla^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial z^2, \beta = \mu/(\lambda + 2\mu)$ is the ratio of the classical shear wave speed to the classical longitudinal wave speed and $\varepsilon = \gamma^2 T_0 / [\rho C_E (\lambda + 2\mu)]$ is defined as the dimensionless thermoelastic coupling constant.

Introducing the displacement potentials ϕ (corresponds to dilatational wave) and ψ (corresponds to shear or transverse wave) through Helmholtz vector decomposition technique as

$$u = \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial z}, \quad w = \frac{\partial \phi}{\partial z} + \frac{\partial \psi}{\partial x}, \tag{28}$$

and plugging it into Equations (25)–(27), we obtain

$$\nabla^2 \phi - \frac{\partial^2 \phi}{\partial t^2} - \Theta = 0, \tag{29}$$

$$\beta \nabla^2 \psi - \frac{\partial^2 \psi}{\partial t^2} = 0, \tag{30}$$

$$\nabla^2 \Theta = \frac{\partial}{\partial t} (1 + \tau D_\tau) \left(\Theta + \varepsilon \nabla^2 \phi \right), \tag{31}$$

Equations (29) and (31) show that the thermal field Θ is coupled with the potential ϕ and so create two quasi-thermal-elastic waves, one of them is called a quasi-elastic wave (qP-wave), while other is called a quasi-thermal wave (qT-wave). Equation (30) creates one transverse or shear type wave (SV-type wave) which is not affected due to the presence of thermal field as well as the MDD.

3. Dispersion equation and its solution

To seek the plane harmonic wave solutions of Equations (29)–(31) propagating in the positive direction of a unit vector **n** with speed c, the solutions of Equation (29)–(31) may be assumed as [28,33]

$$(\phi, \Theta, \psi) = (A_\phi, A_\Theta, A_\psi) \exp\{\iota(k\mathbf{n} \cdot \mathbf{r} - \omega t)\}, \tag{32}$$

where A_ϕ, A_Θ, A_ψ are the constants (possible complex) representing the coefficients of the wave amplitudes, $\iota = \sqrt{-1}$, k is the dimensionless wavenumber, $\mathbf{r}$ $(= x\hat{i} + z\hat{k})$ is the position vector in the x–z plane, $\omega > 0$ is the dimensionless assigned angular frequency. The quantities k and c are connected with ω through the relation $\omega = kc$.

Substituting from Equation (32) into Equations (29)–(31), we get

$$(k^2 - \omega^2)A_\phi + A_\Theta = 0, \tag{33}$$

$$\left(k^2 - \frac{\omega^2}{\beta}\right) A_\psi = 0, \tag{34}$$

$$\iota\varepsilon\omega(1 + G)k^2 A_\phi + \left[k^2 - \iota\omega(1 + G)\right] A_\Theta = 0, \tag{35}$$

where

$$G \equiv G(\tau, \omega) = (A\omega + \iota B)[1 - \exp(\iota\tau\omega)]/\omega - B\tau \exp(\iota\tau\omega). \tag{36}$$

The condition for the existence of non-vanishing solution for A_ϕ and A_Θ of the system of Equations (33) and (35) yields the following dispersion relation

$$k^4 - L_1 k^2 + L_2 = 0, \tag{37}$$

where

$$L_1 = \iota\omega(1 + G)(1 + \varepsilon) + \omega^2, \quad L_2 = i\omega^3(1 + G).$$

The quadratic Equation (37) in k^2 is the general dispersion relation for wave propagation in thermoelastic solid with MDD. Clearly the coefficient L_1 and L_2 are complex for $\omega > 0$. The two roots of (37) and the only root of Equation (34) are given by

$$k_{2,1}^2 = \frac{1}{2}\left[L_1 \pm \sqrt{L_1^2 - 4L_2}\right], \tag{38}$$

$$k_3^2 = \frac{\omega^2}{\beta}. \tag{39}$$

Here k_1^2 corresponds to ' $-$ ' sign and k_2^2 corresponds to '+' sign. Out of the four roots $\pm k_{1,2}$, we consider those two roots only for which $\Im(k_{1,2}) \geq 0$ for the waves to be physically realistic. These two complex wavenumbers give us two distinct types of attenuated and dispersive coupled dilatational waves: one quasi-elastic wave (qP-wave) and one quasi-thermal wave (qT-wave). Besides, since the wavenumbers of both qP- and qT-waves are complex, so they are inhomogeneous waves. The non-dimensional phase speeds, V_j and the attenuation coefficients, Q_j $(j = 1, 2)$ of the qP- and qT-waves can be obtained from the formulae [28,33]

$$V_j = \frac{\omega}{\Re(k_j)}, \quad Q_j = \Im(k_j). \tag{40}$$

Moreover, it is to be noted that, in the presence of attenuation, Equation (32) can be re-written as

$$(\phi, \Theta, \psi) = (A_\phi, A_\Theta, A_\psi) \exp\{-\Im(k)\mathbf{n} \cdot \mathbf{r}\} \exp\{\iota\, (\Re(k)\mathbf{n} \cdot \mathbf{r} - \omega t)\}.$$

Since the coupled thermal-elastic waves are attenuated in the medium considered, the above equation shows that the attenuation direction and the propagation direction may be different in the wave propagation process.

In case of uncoupled thermoelasticity $(\varepsilon = 0)$, we found

$$V_1 = 1, \quad V_2 = \frac{\sqrt{\omega}}{\Re[\iota(1+G)]^{1/2}}. \tag{41}$$

Thus, for $\varepsilon \neq 0$, we conclude that while V_1 represents the speed of the qP-wave, V_2 the speed of the qT-wave (according to our consideration of the sign of k_1^2 and k_2^2). When $\varepsilon \neq 0$, the qP-wave and qT-wave are coupled thermal-elastic waves and the coupling is measured by the following amplitude ratio:

$$\left(\frac{A_\Theta}{A_\phi}\right)_j = (\omega^2 - k_j^2) = \frac{\varepsilon\omega(1+G)k_j^2}{\left[\omega(1+G) + \iota k_j^2\right]} = \zeta_j \; (j = 1, 2). \tag{42}$$

Equation (40) shows that there exist one SV-type wave of wavenumber k_3 which remains unaffected by the thermal wave effect. The phase speed, V_3 and the attenuation coefficient, Q_3 of this wave are

$$V_3 = \sqrt{\beta}, \quad Q_3 = 0, \tag{43}$$

which clearly indicate that the SV-type wave is non-dispersive as well as experience no attenuation.

3.1. Perturbation solution of dispersive waves

The perturbation method has been widely used (Nayfeh and Nemat-Nasser [34], Roychoudhuri [35], Sharma et al. [24]) to study the wave propagation problems in classical (coupled) and non-classical (generalized) thermoelastic continua. Here, our aim is to derive the perturbation solutions for the roots $\pm k_{1,2}$ in this section. The secular Equation (37) can be re-written as

$$f(k^2) - \varepsilon g(k^2) = 0, \tag{44}$$

where

$$f(k^2) = k^4 - k^2\left[\iota\omega(1+G) + \omega^2\right] + i\omega^3(1+G), \quad g(k^2) = \iota\omega(1+G)k^2. \tag{45}$$

For most of the materials, the thermo-mechanical coupling parameter ε is very small and therefore, we develop series expansions in terms of ε for the roots k_j^2 $(j = 1, 2)$ of the Equation (44) in order to explore the effect of various interacting fields on the waves. Thus, for $\varepsilon \ll 1$, we obtain from (44) and (45) that

$$k_1^2(\varepsilon) = \omega^2\left[1 - \frac{(1+G)}{(1+G+\iota\omega)}\varepsilon + \cdots\right], \quad k_2^2(\varepsilon) = \iota\omega(1+G)\left[1 + \frac{(1+G)}{(1+G+\iota\omega)}\varepsilon + \cdots\right]. \tag{46}$$

Using the perturbation solution (46) into Equation (40), the phase speeds and the attenuation coefficients can be obtained for small ε.

4. Reflection phenomenon of thermoelastic waves

In view of the results of the preceding section, we consider an incident qP-wave which is propagating obliquely toward the surface $z = 0$ as in Figure 1. Assuming that the radiation in vacuum is neglected, when it impinges the boundary $z = 0$, three reflected waves in the medium are created. Suppose the reflected qP-, qT- and SV-type waves make angles θ_1, θ_2 and θ_3, respectively, with positive z-axis. Then the complete structure of the wave fields consisting of the incident and reflected waves in the medium Ω may be written as

$$\phi = A_0 \exp\{\iota k_1(x\sin\theta_0 - z\cos\theta_0) - \iota\omega t\} + \sum_{j=1}^{2} A_j \exp\left\{\iota k_j(x\sin\theta_j + z\cos\theta_j) - \iota\omega t\right\}, \tag{47}$$

$$\Theta = \zeta_1 A_0 \exp\{\iota k_1(x\sin\theta_0 - z\cos\theta_0) - \iota\omega t\} + \sum_{j=1}^{2} \zeta_j A_j \exp\left\{\iota k_j(x\sin\theta_j + z\cos\theta_j) - \iota\omega t\right\}, \tag{48}$$

$$\psi = B_1 \exp\{\iota k_3(x\sin\theta_3 + z\cos\theta_3) - \iota\omega t\}. \tag{49}$$

where A_1, A_2 and B_1 represent the coefficients of amplitudes of the reflected qP-, qT- and SV-waves, respectively, and A_0 represents the amplitude coefficient of the incident qP-wave with phase speed V_1. The reflection coefficients are defined as the amplitude ratios of reflected to the incident wave and are determined by the well-defined boundary conditions on the surface $z = 0$.

4.1. Boundary conditions: stress-free isothermal surface

We consider the surface $z = 0$ as stress-free and isothermal. These conditions may be mathematically expressed as follows:

$$\tau_{zz} = \tau_{zx} = \Theta = 0, \quad \text{at} \quad z = 0. \tag{50}$$

In terms of potential functions, the first two conditions in (50) can be written as

$$\left(\frac{\partial^2\phi}{\partial z^2}+\frac{\partial^2\phi}{\partial x^2}\right)+2\beta\left(\frac{\partial^2\psi}{\partial x\partial z}-\frac{\partial^2\phi}{\partial x^2}\right)-\Theta=0, \tag{51}$$

$$\left(2\frac{\partial^2\phi}{\partial x\partial z}+\frac{\partial^2\psi}{\partial x^2}-\frac{\partial^2\psi}{\partial z^2}\right)=0. \tag{52}$$

In order to satisfy the above boundary conditions at $z=0$, the following relations between the angle of the incident wave and the angle of the reflected waves need to be hold on $z=0$:

$$k_1\sin\theta_0=k_1\sin\theta_1=k_2\sin\theta_2=k_3\sin\theta_3, \tag{53}$$

Equation (53) can also be written in the form

$$\theta_0=\theta_1 \text{ and } \frac{\sin\theta_0}{V_1}=\frac{\sin\theta_2}{V_2}=\frac{\sin\theta_3}{V_3}, \tag{54}$$

which is often refereed as *extended Snell's law*.

4.2. Incident qP− wave at the stress-free isothermal boundary

Substituting from Equations (47)–(49) into (50)–(52) and using the relation (53), the following system of equations satisfied by the reflection coefficients $X_1=A_1/A_0$, $X_2=A_2/A_0$, $X_3=B_1/A_0$ of the reflected qP-, qT- and SV-type waves, respectively is obtained:

$$\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\\zeta_1&\zeta_2&0\end{bmatrix}\begin{bmatrix}A_1/A_0\\A_2/A_0\\B_1/A_0\end{bmatrix}=\begin{bmatrix}-a_{11}\\a_{21}\\-\zeta_1\end{bmatrix}, \tag{55}$$

where

$$a_{11}=\omega^2-2\beta k_1^2\sin^2\theta_0,\quad a_{12}=\omega^2-2\beta k_2^2\sin^2\theta_2,\quad a_{13}=\omega^2\sin 2\theta_3,$$
$$a_{21}=k_1^2\sin 2\theta_0,\quad a_{22}=k_2^2\sin 2\theta_2,\quad a_{23}=-k_3^2\cos 2\theta_3.$$

After solving (55), we get the reflection coefficients as follows:

$$X_1=\frac{\begin{array}{c}\zeta_1k_2^2\left(2\beta^2k_3^2\sin^2\theta_2\cos 2\theta_3-\omega^2\sin 2\theta_2\sin 2\theta_3\right)\\+k_3^2\cos 2\theta_3\left[\zeta_2\left(\omega^2-2\beta^2k_1^2\sin^2\theta_0\right)-\zeta_1\omega^2\right]+\zeta_2k_1^2\left(-\omega^2\right)\sin 2\theta_0\sin 2\theta_3\end{array}}{\begin{array}{c}\zeta_1k_2^2\left(\omega^2\sin 2\theta_2\sin 2\theta_3\right)-2\beta^2k_3^2\sin^2\theta_2\cos 2\theta_3\right)\\+k_3^2\cos 2\theta_3\left[\zeta_1\omega^2-\zeta_2\left(\omega^2-2\beta^2k_1^2\sin^2\theta_0\right)\right]-\zeta_2k_1^2\omega^2\sin 2\theta_0\sin 2\theta_3\end{array}}, \tag{56}$$

$$X_2=-\frac{2\zeta_1k_1^2\omega^2\sin 2\theta_0\sin 2\theta_3}{\begin{array}{c}\zeta_1k_2^2\left(2\beta^2k_3^2\sin^2\theta_2\cos 2\theta_3-\omega^2\sin 2\theta_2\sin 2\theta_3\right)\\+k_3^2\cos 2\theta_3\left[\zeta_2\left(\omega^2-2\beta^2k_1^2\sin^2\theta_0\right)-\zeta_1\omega^2\right]+\zeta_2k_1^2\omega^2\sin 2\theta_0\sin 2\theta_3\end{array}}, \tag{57}$$

$$X_3=\frac{2k_1^2\sin 2\theta_0\sec 2\theta_3\left[\zeta_1\left(\omega^2-2\beta^2k_2^2\sin^2\theta_2\right)-\zeta_2\left(\omega^2-2\beta^2k_1^2\sin^2\theta_0\right)\right]}{\begin{array}{c}k_3^2\left[\zeta_2\left(\omega^2-2\beta^2k_1^2\sin^2\theta_0\right)-\zeta_1\left(\omega^2-2\beta^2k_2^2\sin^2\theta_2\right)\right]\\+\omega^2\tan 2\theta_3\left(\zeta_2k_1^2\sin 2\theta_0-\zeta_1k_2^2\sin 2\theta_2\right))\end{array}}. \tag{58}$$

These expressions exhibit that the reflection coefficients depend on the angle of incidence (θ_0), time-delay parameter (τ) of MDD and on the material properties of the thermoelastic medium Ω.

4.3. Remarks

For uncoupled thermoelasticity, we put $\varepsilon = 0$ which gives $\zeta_j = 0$ $(j = 1, 2)$. Hence, there will be no reflected qT-wave in this case. Consequently, $X_2 = 0$ at all angle of incidence θ_0.

5. Numerical results and discussions

In this section, we perform some numerical results in order to illustrate the analytical results calculated in the previous sections for the reflection coefficients, phase speeds and attenuation coefficients. For this purpose, copper like material is modeled as the thermoelastic material for which the following values of the different physical constants are borrowed from [16]:

Using the *MATLAB* software, the variations of the absolute values of the reflection coefficients X_j $(j = 1, 2, 3)$ with respect to the angle of incidence θ_0 are presented graphically through Figures 2–6 for the incident qP-wave at the isothermal stress-free surface $z = 0$. Using the numerical values given in Table 1, numerically computed values of the reflection coefficients are plotted with θ_0 for the range $0^\circ \leq \theta_0 \leq 90^\circ$.

In present work, we devoted to investigate the reflection phenomena of thermoelastic waves at a stress-free isothermal boundary by considering the memory dependence for heat transfer with thermal relaxation. The key parameters are the time-delay factor and the kernel function of the MDD and the thermoelastic coupling parameter ε. As described in the manuscript, the most advantage of the generalized thermoelastic model based on heat

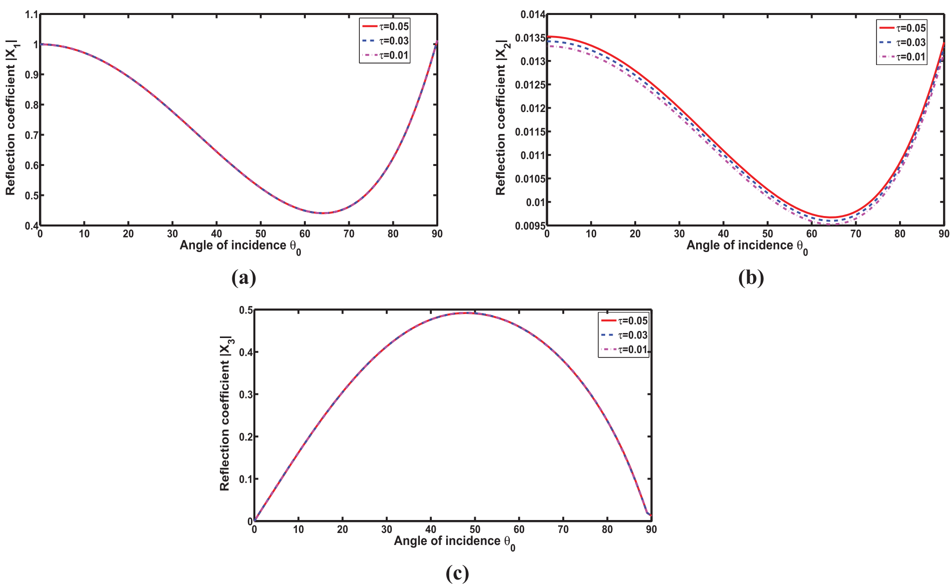

Figure 2. Variations of $|X_j|$ vs. θ_0 for different delay times τ when $K(t - \xi) = 1/2 - (t - \xi)/\tau$.

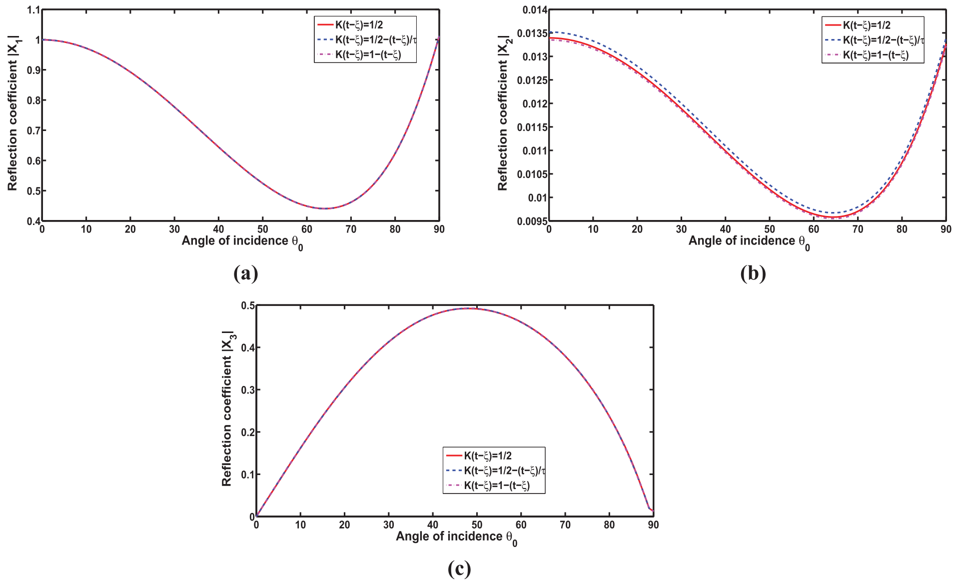

Figure 3. Variations of $|X_j|$ vs. θ_0 for different kernel function, $K(t-\xi)$ when $\tau = 0.05$.

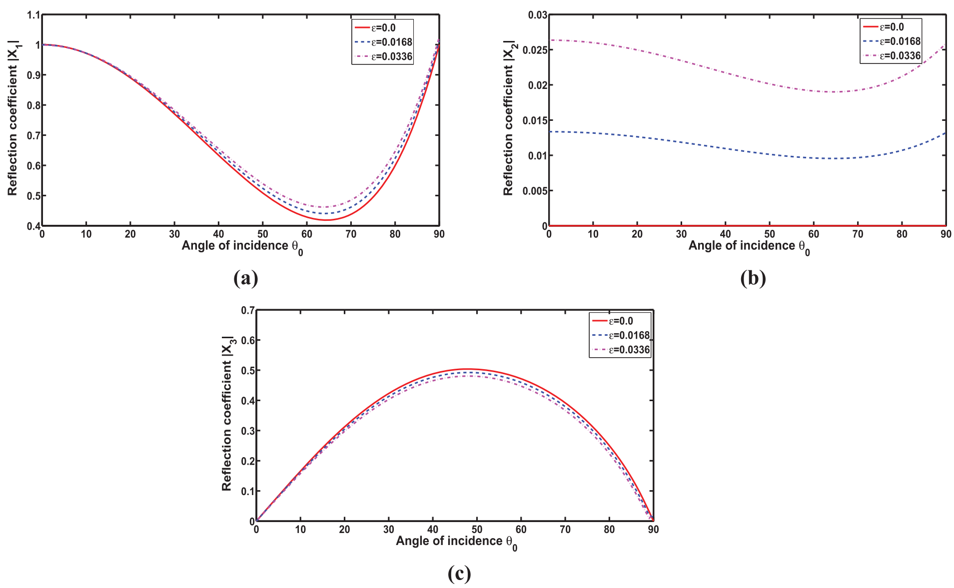

Figure 4. Variations of $|X_j|$ vs. θ_0 for different ε when $\tau = 0.05, K(t, \xi) = 1 - (t - \xi)$.

transfer with MDD is the free choosing of the delay time factor and the kernel function, which makes it flexible in applications. In the obtained numerical results, we demonstrated the different effect of the time-delay factor, kernel function and thermoelastic coupling parameter on the variations of the reflection coefficients of the reflected qP-, qT- and SV-type wave as follows:

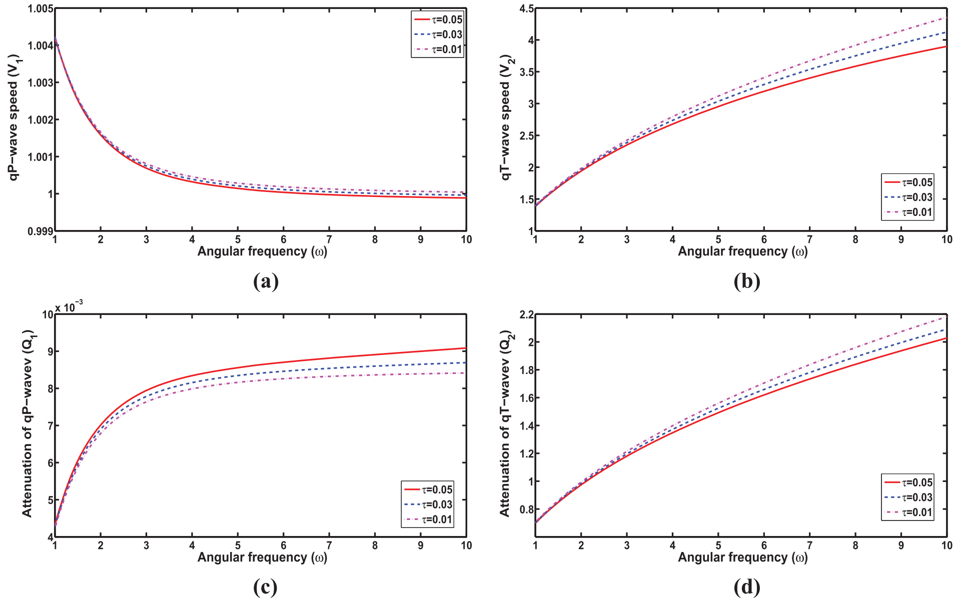

Figure 5. Effect of τ on the phase speeds, V_j (dimensionless) and the corresponding attenuation coefficients Q_j $(j = 1, 2)$ (dimensionless) vs. non-dimensional angular frequency ω when $K(t, \xi) = 1 - (t - \xi)/\tau$.

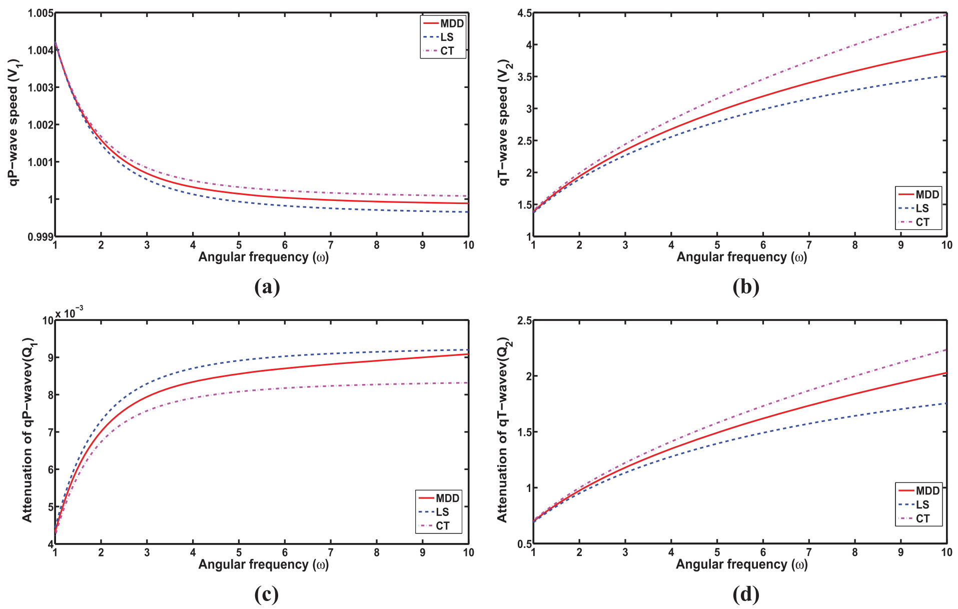

Figure 6. Comparison of V_j (dimensionless) and Q_j (dimensionless) for the MDD, LS and CT theories with respect to ω (dimensionless) for fixed $K(t, \xi) = 1 - (t - \xi)/\tau$.

Figure 2(a–c) shows the profiles of the reflection coefficients, $|X_j|$ with respect to the angle of incidence θ_0 for three values of the delay time τ, namely $\tau = 0.01, 0.03, 0.05$ when $K(t, \xi) = 1/2 - (t - \xi)/\tau$. It is evident from Figure 2(a) that the reflection coefficient $|X_1|$ of

Table 1. Numerical values of the material constants.

Symbol	Value	Unit	Symbol	Value	Unit
λ	7.76×10^{10}	N/m^2	μ	3.86×10^{10}	N/m^2
T_0	293	K	ρ	8954	kg/m^3
C_e	383.1	m^2/K	K_T	386	N/K s
α_T	383.1	K^{-1}	ε	0.0168	–

the reflected qP-wave decreases for $0^\circ \leq \theta_0 \leq 65^\circ$ and then increases in $65^\circ \leq \theta_0 \leq 90^\circ$ to reach unity for all values of τ. The maximum of $|X_1|$ is unity which occurs at $\theta_0 = 0^\circ$ and 90°. Figure 2(b) exhibits that the reflection coefficient $|X_2|$ of the reflected qT-wave attains its maximum at $\theta_0 = 0^\circ, 90^\circ$. From Figure 2(c), we observed that the reflection coefficient $|X_3|$ of the reflected SV-type wave first increases in the range $0^\circ \leq \theta_0 \leq 48^\circ$ and then decreases for $48^\circ \leq \theta_0 \leq 90^\circ$ for all τ. It is maximum near $\theta_0 = 48^\circ$ and vanishes at $\theta_0 = 0^\circ$. Figure 3(a–c) reveals the profiles of $|X_j|$ with θ_0 for three different kernel function, namely $K(t-\xi) = 1/2,\ 1/2 - (t-\xi)/\tau$ and $1-(t-\xi)$ when $\tau = 0.05$. It is evident from these figures that the qualitative behaviors of the reflection coefficients, $|X_j|$ are similar to those presented in Figure 2(a–c), respectively.

The key point, noticed from Figures 2(a–c) and 3(a–c) is that while no significant effects of the delay time factor and the kernel function on the variations of $|X_1|$ and $|X_3|$ can be seen, only significant influences of τ and $K(t-\xi)$ can be seen markedly on the variation of $|X_2|$. This is most probably due to the following mechanism: as shown in Equation (13) or (14), the MDD are introduced directly into the heat conduction equation instead of the constitutive equation to characterize the effects of the delay time parameter and the kernel function on the reflection coefficients of the various reflected waves, which in turn leads to the consequence that the delay time parameter and the kernel function barely influences the reflection coefficient, $|X_2|$ of the reflected qT-wave.

Figure 4(a–c) is drawn to analyze the influence of the thermoelastic coupling parameter ε on the profiles of $|X_j|$ at fixed $K(t,\xi) = 1-(t-\xi),\ \tau = 0.05$. Here, we take three values of ε as 0, 0.0168 and 0.0336. We see from these figures that the absolute values of X_1 and X_2 have large value for large ε, meaning it has an increasing effect on $|X_1|$ and $|X_2|$, while it has a decreasing effect on $|X_3|$. We can also observe from these figures that the influence of the coupling parameter on $|X_1|$ and $|X_2|$ is very small as compared to that of $|X_3|$. Another interesting fact is revealed in Figure 3(b) that the reflection coefficient X_2 vanishes identically at each θ_0 for $\varepsilon = 0$ (uncoupled thermoelasticity) which is in complete agreement with our analytical results obtained in subsection 4.3.

Figures 2–4 reveal that the reflection coefficient $|X_2|$ of reflected qT-wave is very small as compared to the absolute values of the reflection coefficients of the reflected qP- and SV-type waves. Thus the energy carried along the reflected qT-wave is the least which in turn means that maximum amount of the incident energy is carried along the reflected qP- and the SV-type waves.

The dependence of the dimensionless phase speeds, V_j and the corresponding attenuation coefficients, Q_j $(j = 1, 2)$ of the qP- and qT-waves, respectively, on the dimensionless angular frequency ω is expressed through Figure 5(a–d) with the variation of τ. Following Chadwick and Sneddon [36], we have selected the range of the dimensionless ω as $1 \leq \omega \leq 10$. Figure 5(a,b) depict that the phase speed, V_1 of the qP-wave is smaller than the phase speed, V_2 of the qT-wave in the entire range of ω. The phase speed V_1 decreases while V_2 increases as ω increases for all values of τ. Figure 5(c,d) exhibits that the qP-wave is less attenuated when compared to the attenuation of the qT-wave. From these figures, it appears that the qP- and qT-waves are dispersive in nature as well as attenuated which are the verification of the analytical results pointed out in the text in Section 3.

Figure 6(a–d) is drawn in order to compare V_j and Q_j for the MDD, Lord-Shulman (LS) and the coupled thermoelasticity (CT) [32] theories. It is observed from Figure 6(a,b) that the values of V_j are found to be larger for the CT theory while that are found to be smaller for the LS theory. Similar patterns are noticed for the corresponding attenuation coefficients in Figure 6(c,d) for all the theories. It is also clear from the graphs that the phase speeds and the attenuation coefficients of the qP- and qT-waves, respectively, reveal qualitatively similar nature for the MDD, LS and CT theories, however, dissimilarity lies on the ground of numerical values.

6. Conclusions

In this manuscript, the significance of the selected kernel and the time-delay parameter of MDD on plane harmonic wave propagation in a homogeneous, isotropic solid conducting heat, has been studied by applying the new linear theory of generalized thermoelasticity based on the heat conduction with MDD. The following points can be concluded according to the analysis above:

(1) The reflection coefficients depend on the angle of incidence, MDD and the thermoelastic coupling parameter.
(2) The time-delay and the kernel function markedly affect the reflection coefficients of the reflected qT-wave only.
(3) The reflection coefficient of the reflected qT-wave is highly influenced by the thermoelastic coupling parameter as compared to the others.
(4) The time-delay reveals a strong effect on the phase speeds and the attenuation coefficients of the qP- and qT-waves. The phase speed and the attenuation coefficient of the SV-type wave is independent of the time-delay and the kernel function.
(5) It is observed that the present new theory of thermoelasicity with MDD supports the finite speed of the thermal wave (qT-wave) propagation through the medium. Hence, this theory is indeed a generalized theory of thermoelasticity. The present work is very much expected to be useful for investigating various wave propagation problems: both theoretically and in observation of wave propagation. In particular, the present work is of geophysical interest for investigations on earthquakes and similar phenomena in seismology and engineering where 'MDD may play a significant role'. We also hope that our present theoretical results may provide some useful information for experimental scientists, researchers, and seismologists working on wave propagation problems in thermoelastic solid.

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Disclosure statement

No potential conflict of interest was reported by the author(s).

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